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A hybrid model for optimal decisions within personal finance and retirement

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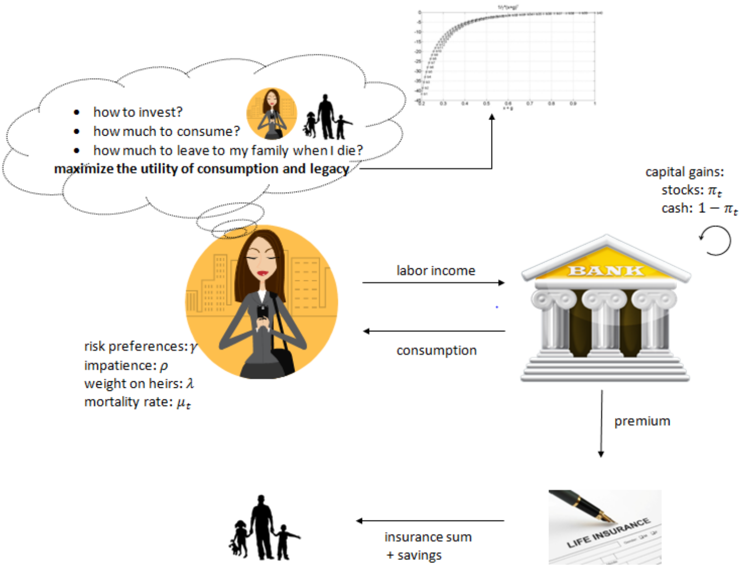
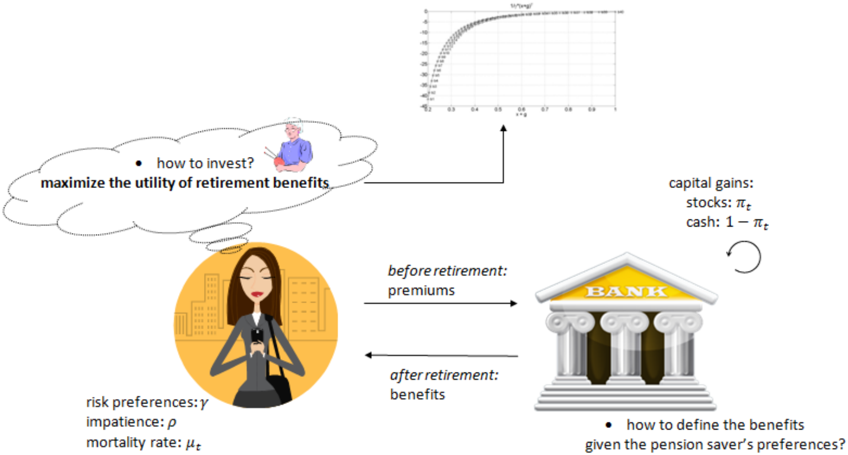
case (A) - optimal investment, consumption and life insurance, [Richard, 1975]



case (B) - optimal investment with optimal annuities



Optimization technology

stochastic optimal control - explicit solutions

✓ ideal framework - produce an optimal policy that is easy to understand and implement

✗ explicit solution may not exist

✗ can't deal with details

stochastic (linear) programming (SLP)

✓ general purpose decision model with an objective function that can take a wide variety of forms

✓ can address realistic considerations, such as transaction costs

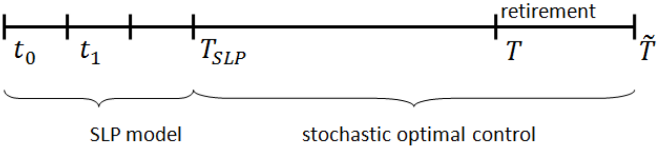
✓ can deal with details

✗ problem size grows quickly as a function of number of periods and scenarios

✗ challenge to select a representative set of scenarios for the model

Hybrid model

- first years decisions - multi-stage stochastic linear programming **(SLP)**
- decisions for the long steady period - **stochastic optimal control** (dynamic programming)



Dynamic programming I

Wealth dynamics

$$dX_t = \big(r + \pi_t(\alpha - r)\big)X_t dt + \pi_t \sigma X_t dW_t + I_t dt - c_t dt - \mu_t^* I_t dt,$$
$$X_0 = x_0.$$

Maximize expected utility of consumption and bequest

$$V(t,x) = \sup_{\pi,c,I \in \mathcal{Q}[t,\widetilde{T})} E_{t,x}\left[\int_t^{\widetilde{T}} e^{-\int_t^s \mu_\tau d\tau}\Big(u(s,c) + \mu_s U(s, X_s + I_s)\Big)ds\right],$$

with the utility functions:

$$u(c,t) = \frac{1}{\gamma} w^{1-\gamma}(t) c^\gamma = \frac{1}{\gamma} e^{-\rho t} c^\gamma, \qquad U(x) = \frac{1}{\gamma} v^{1-\gamma}(t) x^\gamma = \frac{1}{\gamma} \lambda^{-\gamma} e^{-\rho t} x^\gamma,$$

$1-\gamma$ - risk aversion, ρ - impatience factor, λ - weight on bequest, μ_t - mortality rate, μ_t^* - pricing mortality rate.

Dynamic programming II

Solution (optimal value function)

$$V(t,x) = \tfrac{1}{\gamma} f^{1-\gamma}(t)\big(x+g(t)\big)^{\gamma},$$

$$f(t) = \int_t^{\widetilde{T}} e^{-\frac{1}{1-\gamma}\int_t^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right) d\tau} \left[w(s) + \left(\frac{\mu_s}{(\mu_s^*)^\gamma}\right)^{1/(1-\gamma)} v(s)\right] ds,$$

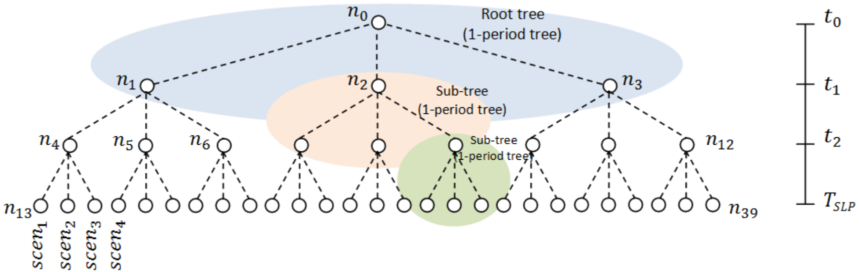
$$g(t) = \int_t^{\widetilde{T}} e^{-\int_t^s (r+\mu_\tau^*) d\tau} l(s) ds.$$

The optimal controls

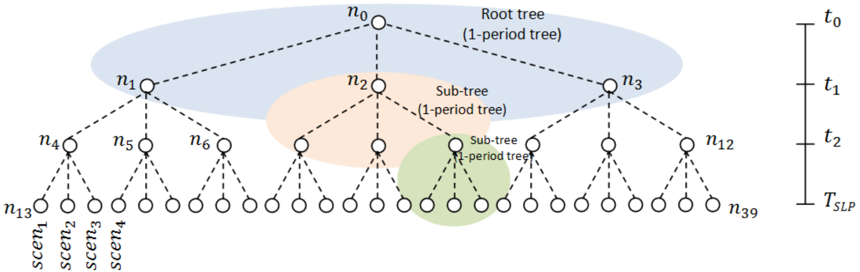
$$\pi_t^* = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g(t)}{X_t}, \qquad c_t^* = \frac{w(t)}{f(t)}\big(X_t + g(t)\big),$$

$$l_t^* = \left(\frac{\mu_t}{\mu_t^*}\right)^{1/(1-\gamma)} \frac{v(t)}{f(t)} \left(X_t + g(t)\right) - X_t.$$

SLP model - objective

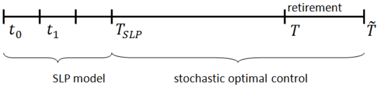


SLP model - objective

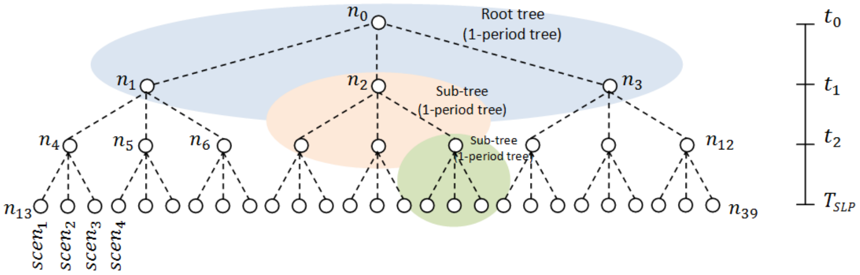


$$\sum_{s=t_0}^{T_{SLP}-1} \sum_{n_{\nu(s)}} Pr_{n_{\nu(s)}} \cdot e^{-\int_{t_0}^{s} \mu_\tau d\tau} \left[u(s, \widetilde{C}_{n_{\nu(s)}}) + \mu_s U(s, \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(s)}} + \widetilde{I}_{n_{\nu(s)}}) \right]$$

$$+ \sum_{n_{\nu(T_{SLP})}} Pr_{n_{\nu(T_{SLP})}} \cdot e^{-\int_{t_0}^{T_{SLP}} \mu_\tau d\tau} \cdot V\left(T_{SLP}, \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(T_{SLP})}}\right) \longrightarrow \max$$

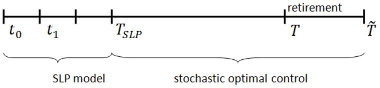


SLP model - objective



$$\sum_{s=t_0}^{T_{SLP}-1} \sum_{n_{\nu(s)}} Pr_{n_{\nu(s)}} \cdot e^{-\int_{t_0}^{s} \mu_\tau d\tau} \left[u(s, \widetilde{C}_{n_{\nu(s)}}) + \mu_s U(s, \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(s)}} + \widetilde{I}_{n_{\nu(s)}}) \right]$$

$$+ \sum_{n_{\nu(T_{SLP})}} Pr_{n_{\nu(T_{SLP})}} \cdot e^{-\int_{t_0}^{T_{SLP}} \mu_\tau d\tau} \cdot V\left(T_{SLP}, \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(T_{SLP})}}\right) \longrightarrow \max$$



- **Obs!** linearize the objective

Constraints

budget equation, $t = t_0, \dots, T_{SLP} - 1$ and $\nu(t) = 1, \dots, K_t$,

$$\sum_{i=1}^{N} \widetilde{P}^i_{n_\nu(t)} + \widetilde{C}_{n_\nu(t)} + \mu^*_t \widetilde{I}_{n_\nu(t)} = x_0 \mathbb{1}_{\{t=t_0\}} + \sum_{i=1}^{N} \widetilde{S}^i_{n_\nu(t)} + I_t, \tag{1}$$

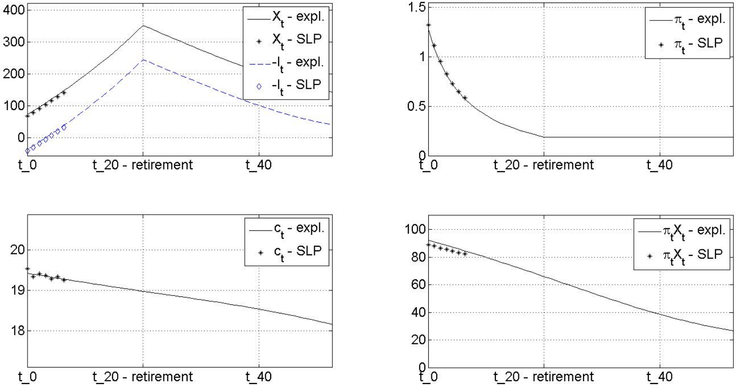
asset inventory balance, $t = t_0, \dots, T_{SLP}$, $\nu(t) = 1, \dots, K_t$, $i = 1, \dots, N$:

$$\widetilde{X}^i_{n_{\nu(t)}} = \left(1 + R^i_{n_{\nu(t)}}\right) \widetilde{X}^i_{n_{\nu(t-1)}} \mathbb{1}_{\{t>t_0\}} + \widetilde{P}^i_{n_{\nu(t)}} \mathbb{1}_{\{t<T_{SLP}\}} - \widetilde{S}^i_{n_{\nu(t)}} \mathbb{1}_{\{t<T_{SLP}\}}, \tag{2}$$

non-negativity, $t = t_0, \dots, T_{SLP} - 1$, $\nu(t) = 1, \dots, K_t$:

$$\widetilde{C}_{n_{\nu(t)}} > 0, \qquad \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(t)}} + \widetilde{I}_{n_{\nu(t)}} > 0, \qquad \widetilde{P}^i_{n_{\nu(t)}} \geq 0, \qquad \widetilde{S}^i_{n_{\nu(t)}} \geq 0, \tag{3}$$

Results I - case (A)



- payments: $l_t = 27.000$ EUR, $x_0 = 60.000$ EUR,
- market: $N = 2$, $r = 0.02$, $\alpha = 0.04$, $\sigma = 0.2$,
- utility function: $\gamma = -3$, $\rho = 0.04$, $\lambda = 10$,
- $age_0 = 45$, $age_T = 65$,
- life uncertainty: $\theta = 0.0$, $\beta = 4.59364$, $\delta = 0.05032$,
- scenario tree: $T_{SLP} = 8$, $bf = 3$, number of trees $= 10$,
- linearization: $m = 40$,
  $bp_1^c = 0.7E[c^*_{T_{SLP}}]$, $bp_m^c = 2E[c^*_{T_{SLP}}]$,
  $bp_1^{x+g} = 0.5E[X^*_{T_{SLP}} + g_{T_{SLP}}]$, $bp_m^{x+g} = 2E[X^*_{T_{SLP}} + g_{T_{SLP}}]$,
  $bp_1^{x+ins} = 0.5E[X^*_{T_{SLP}} + I^*_{T_{SLP}}]$, $bp_m^{x+ins} = 1.5E[X^*_{T_{SLP}} + I^*_{T_{SLP}}]$.

Modified constraints

budget equation with transaction costs, $t = t_0, \ldots, T_{SLP} - 1,\ \nu(t) = 1, \ldots, K_t$

$$\sum_{i=1}^{N} \widetilde{P}^i_{n_\nu(t)}(1 + q^i) + \widetilde{C}_{n_\nu(t)} + \mu^*_t \widetilde{I}_{n_\nu(t)} = x_0 \mathbb{1}_{\{t=t_0\}} + \sum_{i=1}^{N} \widetilde{S}^i_{n_\nu(t)}(1 - q^i) + I_t, \tag{1'}$$

asset inventory balance with taxes on capital gains, $t = t_0, \ldots, T_{SLP},\ \nu(t) = 1, \ldots, K_t$, $i = 1, \ldots, N$:

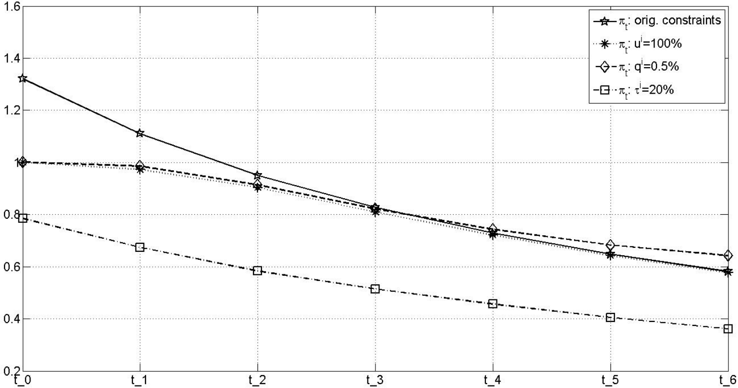
$$\widetilde{X}^i_{n_{\nu(t)}} = \left(1 + {}_{net}R^i_{n_{\nu(t)}}\right)\widetilde{X}^i_{n_{\nu(t-1)}} \mathbb{1}_{\{t>t_0\}} + \widetilde{P}^i_{n_{\nu(t)}} \mathbb{1}_{\{t<T_{SLP}\}} - \widetilde{S}^i_{n_{\nu(t)}} \mathbb{1}_{\{t<T_{SLP}\}}, \tag{2'}$$

limits on portfolio composition, $t = t_0, \ldots, T_{SLP},\ \nu(t) = 1, \ldots, K_t$:

$$\widetilde{X}^i_{n_{\nu(t)}} \geq d_i \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(t)}}, \quad \widetilde{X}^i_{n_{\nu(t)}} \leq u_i \sum_{i=1}^{N} \widetilde{X}^i_{n_{\nu(t)}}. \tag{4}$$

Results II - case (A) with modifications

Optimal investment: a) original constraints, b) limit on portfolio composition, $u = 100\%$, c) transaction costs, $q = 0.5\%$, d) taxes on capital gains, $\tau = 20\%$.



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